Volume of a Sphere

Below we prove that the volume of Sphere S with radius r is given by the equation: $V_S = \frac{4}{3}\pi r^3$.

Proof using Multi-Variate Integration

A Sphere S with radius r and centre the origin, has the Cartesian equation:

$$x^2 + y^2 + z^2 = r^2$$

Hence, S can be described as the set of points:

$$S = \left\{(x, y, z) : -r \leq x \leq r,\ -\sqrt{r^2 - x^2} \leq y \leq \sqrt{r^2 - x^2},\ -\sqrt{r^2 - x^2 - y^2} \leq z \leq \sqrt{r^2 - x^2 - y^2}\right\}$$

Then the Volume of S can be found:

$$V_S = \int_S dV \quad \text{where } dV = dx\, dy\, dz.$$

To simplify the integral, we convert to spherical polar co-ordinates, where S can be defined as the set of points:

$$S = \left\{(\rho, \phi, \theta) : 0 \leq \rho \leq r,\ 0 \leq \phi \leq 2\pi,\ 0 \leq \theta \leq \pi\right\}.$$

Since $dV = \rho^2 \sin\theta \ d\rho\, d\phi\, d\theta$ for a sphere, the volume of S can be found:

$$\begin{aligned}
V_S &= \int_0^{\pi} \int_0^{2\pi} \int_0^{r} \rho^2 \sin\theta \ d\rho\, d\phi\, d\theta \\
&= \int_0^{\pi} \sin\theta \ d\theta \cdot \int_0^{2\pi} d\phi \cdot \int_0^{r} \rho^2 \ d\rho \\
&= \Big[-\cos\theta\Big]_0^{\pi} \cdot \Big[\phi\Big]_0^{2\pi} \cdot \left[\frac{\rho^3}{3}\right]_0^{r} \\
&= -\Big[\cos\pi - \cos 0\Big] \cdot 2\pi \cdot \frac{r^3}{3} \\
&= 2 \cdot 2\pi \cdot \frac{r^3}{3} \\
&= \frac{4}{3}\pi r^3.
\end{aligned}$$

$\square$