Volume of a Cylinder

Below we prove that the volume of Cylinder C with radius r and height H is given by the equation: $V_C = \pi r^2 H$.

Proof using Multi-Variate Integration

A Cylinder C with radius r and height H can be described in cylindrical polar co-ordinates as the set of points:

$$C = \big\{(\rho, \phi, z) : 0 \leq \rho \leq r,\, 0 \leq \phi \leq 2\pi,\, 0 \leq z \leq H\big\}.$$

Since $dV = \rho \; d\rho \, d\phi \, dz$ for a cylinder, the volume of C can be found:

$$\begin{aligned} V_C &= \int_0^H \int_0^{2\pi} \int_0^r \rho \; d\rho \, d\phi \, dz \\ &= \int_0^H dz \cdot \int_0^{2\pi} d\phi \cdot \int_0^r \rho \; d\rho \\ &= \Big[z\Big]_0^H \cdot \Big[\phi\Big]_0^{2\pi} \cdot \Big[\frac{\rho^2}{2}\Big]_0^r \\ &= H \cdot 2\pi \cdot \frac{r^2}{2} \\ &= \pi r^2 H. \end{aligned}$$

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