Sum of the squares of the first n integers

Below we prove using Induction that the sum of the squares of the integers from 1 to n is given by:

$$S_n = \sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof By Induction

Using the principle of Induction, we prove the statement for a base case and then prove the inductive hypothesis.

For the base case of n = 1, we have:

$$S_1 = \frac{1 \cdot (1+1) \cdot (2 \times 1+1)}{6} = 1.$$

We now can prove our hypothesis for a fixed $n \in \mathbb{N}$ using the inductive step that:

$$S_{n+1} = S_n + (n+1)^2.$$

Expanding the right hand side of this expression, we see:

$$\begin{aligned} \mathsf{RHS} &= S_n + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{2n^3 + n^2 + 2n^2 + n}{6} + \frac{6n^2 + 12n + 6}{6} \\ &= \frac{2n^3 + 9n^2 + 13n + 6}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= S_{n+1} \\ &= \mathsf{LHS}. \end{aligned}$$

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