## Sum of the squares of the first $n$ integers

Below we prove using Induction that the sum of the squares of the integers from 1 to $n$ is given by:

$$
S_{n}=\sum_{i=0}^{n} i^{2}=\frac{n(n+1)(2 n+1)}{6} .
$$

## Proof By Induction

Using the principle of Induction, we prove the statement for a base case and then prove the inductive hypothesis.
For the base case of $n=1$, we have:

$$
S_{1}=\frac{1 \cdot(1+1) \cdot(2 \times 1+1)}{6}=1 .
$$

We now can prove our hypothesis for a fixed $n \in \mathbb{N}$ using the inductive step that:

$$
S_{n+1}=S_{n}+(n+1)^{2}
$$

Expanding the right hand side of this expression, we see:

$$
\begin{aligned}
\mathrm{RHS} & =S_{n}+(n+1)^{2} \\
& =\frac{n(n+1)(2 n+1)}{6}+(n+1)^{2} \\
& =\frac{2 n^{3}+n^{2}+2 n^{2}+n}{6}+\frac{6 n^{2}+12 n+6}{6} \\
& =\frac{2 n^{3}+9 n^{2}+13 n+6}{6} \\
& =\frac{(n+1)(n+2)(2 n+3)}{6} \\
& =S_{n+1} \\
& =\text { LHS } .
\end{aligned}
$$

