$\sqrt{2}$ is irrational

Below we prove that the $\sqrt{2}$ is irrational, that is, it can't be expressed in the form of a fraction.

Proof By Contradiction

Suppose that $\sqrt{2}$ was rational. That is, there exists two whole numbers $m, n \in \mathbb{N}$ such that:

$$\sqrt{2} = \frac{m}{n}.$$

We can assume that m and n are the smallest such numbers (i.e. the fraction is in it's simpliest form). We note that this means that not both m and n can be even (otherwise the fraction could be simplified).

Then it could be said that:

$$2 = \frac{m^2}{n^2} \implies m^2 = 2n^2.$$

This implies that 2 divides m^2, in other words, m^2 is even. If m^2 is even, then so must m. So there must exist a whole number k such that $2k = m$. Substituting into our equation, we have:

$$\begin{aligned} m^2 = 2n^2 &\implies (2k)^2 = 2n^2 \\ &\implies 4k^2 = 2n^2 \\ &\implies 2k^2 = n^2 \end{aligned}$$

Now we can see that 2 divides n^2 and so by the same argument, n must also even. However, we started with the assumption that not both m and n are even. Hence we have a contraction and it follows that

$$\sqrt{2} \neq \frac{m}{n}, \quad \forall m, n \in \mathbb{N}$$

. $\square$